Tutorial Sheet 6: Group Theory

(1) Give two reasons why the set of odd integers under addition is not a group.

(2) Show that $\{1, 2, 3\}$ under multiplication modulo 4 is not a group but that $\{1, 2, 3, 4\}$ under multiplication modulo 5 is a group.

(3) Prove that in a group, $(a^{-1})^{-1} = a$ for all a.

(4) Prove that a group G is Abelian if and only if $(ab)^{-1} = a^{-1}b^{-1}$ for all a and b in G.

(5) Let p be a prime number and G be a group such that $|G| = p$. The group G is cyclic.

(6) Is D_3 (the set of symmetries of an equilateral triangle) Abelian?

(7) Prove that a group G is Abelian if and only if $(ab)^2 = a^2b^2$ for all a and b in G.

(8) Prove that an Abelian group with two elements of order 2 must have a subgroup of order 4.

(9) Let $\Phi : G \to H$ be a homomorphism. Show that $\Phi(e_G) = e_H$ and $\Phi(a^{-1}) = (\Phi(a))^{-1}$.

(10) Show that A_3 is a normal subgroup of S_3.

(11) Prove or disprove that (a) Every normal subgroup of a group G is cyclic.
(b) Every cyclic subgroup is normal.

(12) Show that the set $Z[x]$ of all polynomials in the variable x with integer coefficients under ordinary addition and multiplication is a commutative ring with unity $f(x) = 1$.

(13) Show that $2\mathbb{Z} \cup 3\mathbb{Z}$ is not a subring of $\mathbb{Z}$.

(14) The ring $\{0, 2, 4, 6, 8\}$ under addition and multiplication modulo 10 has a unity (multiplicative identity). Find it.

(15) Let R be a ring with unity 1. Show that $S = \{n.1 : n \in \mathbb{Z}\}$ is a sub- ring of R.

(16) Suppose that a belongs to a ring and $a^4 = a^2$. Prove that $a^{2n} = a^2$ for all $n \geq 1$.

(17) Let F be a finite field with n elements. Prove that $x^{n-1} = 1$ for all nonzero x in F, 1 denotes the multiplicative identity of F.